Practice Problems

Problems in the final exam can be significantly different from the following problem set.

1. Let $X \sim U(0, 1)$. (i) Calculate $P(X)$. (ii) Find the value of the constant $c$ for which $P(X - c)^2$ is as small as possible. (iii) Find the density of $Y = 1/X$. (iv) Find the density of $Y = \tan(\pi X - \frac{\pi}{2})$.

2. For events $A_1, A_2, \ldots$ that are not necessarily disjoint, establish the inequality
   
   $P(\bigcup_{i=1}^n A_i) \leq \sum_{i=1}^n P(A_i)$

   by the following method: explain why $I_{\bigcup_{i=1}^n A_i} \leq \sum_{i=1}^n I_{A_i}$ and then take expectation to complete the proof, where the indicator function $I_A(\omega) = 1$, for $\omega \in A$; 0, otherwise.

3. Suppose $X$ and $Y$ are independent random variables, each $Uniform(0, 1)$ distributed. Treat $X$ and $Y$ as the coordinates of a point in the unit square. (i) Show that $\text{cov}(X + Y, X - Y) = 0$. (ii) Find the region of the square where $X + Y \geq 7/4$ and $X - Y \geq 3/4$. (iii) Find $P\{X + Y \geq 7/4 | X - Y \geq 3/4\}$. (iv) Show that $X + Y$ and $X - Y$ are not independent.

4. Suppose I have three coins in my pocket: the first lands heads with probability $0.1$, the second with probability $0.5$, and the third with probability $0.9$. I select a coin at random from my pocket and toss it twice. Let $C_i$ denote the event that I choose coin $i$, for $i = 1, 2, 3$, and $H_n$ denote the event that the $n$th toss lands heads, for $n = 1, 2$. (i) Find $P(C_i|H_1)$ for $i = 1, 2, 3$. (ii) Find $P(H_1)$. (iii) Find $P(C_i|H_1)$ for $i = 1, 2, 3$. (iv) Find $P(H_2|H_1)$.

5. A radioactive material emits -particles at a rate described by the density function
   
   $f(t) = \lambda e^{-\lambda t}$, $\lambda$ is a known constant.

   Find the probability that a particle is emitted in the first 10 seconds, given that (i) no particle is emitted in the first second. (ii) no particle is emitted in the first 5 seconds. (iii) a particle is emitted in the first 3 seconds. (iv) a particle is emitted in the first 20 seconds.

6. Let $X$ have a $Bin(n, p)$ distribution. Put $b(k) = P\{X = k\}$, for $k = 0, 1, \ldots, n$. (i) Find the ratio $b(k)/b(k-1)$ for $k = 1, 2, \ldots, n$. (ii) Show that the ratio $b(k)/b(k-1)$ is $\geq 1$ if and only if $1 \leq k \leq (n + 1)p$. (iii) Show that $b(k)$ achieves its maximum value at $k^* = \lfloor (n + 1)p \rfloor$, the largest integer $\leq (n + 1)p$. Show that $b(k)$ increases monotonely for $k \leq k^*$ then decreases monotonely for $k > k^*$. (That is, show that the mode of the Binomial distribution is close to $np$.)
7. For 1-dimensional random walk with $p = 2/3$, show that the probability that the walk never returns to 0 is positive.

8. Suppose 100 envelopes are prepared for 100 letters, but, through an unfortunate accident, the letters are assigned at random (all permutations equally likely), one letter per envelope. Say that letters $i$ and $j$ are switched if letter $i$ is placed in envelope $j$ and letter $j$ is placed in envelope $i$. (i) For each pair $i < j$, define $X_{ij} = 1$, if letters $i$ and $j$ are switched; 0, otherwise. Find $P[X_{ij} = 1]$. (ii) Are the random variables $X_{12}$ and $X_{34}$ independent? Explain. (iii) Find the expected number of switches.

9. Assume that, every time you buy a box of Wheaties, you receive a picture of one of the $n$ players for the New York Yankees (see Exercise 3.2.34). Let $X_k$ be the number of additional boxes you have to buy, after you have obtained $k - 1$ different pictures, in order to obtain the next new picture. Thus $X_1 = 1$, $X_2$ is the number of boxes bought after this to obtain a picture different from the first pictured obtained, and so forth. (i) Show that $X_k$ has a geometric distribution with $p = (n - k + 1)/n$. (ii) Calculate the expected number of boxes to get all players.

10. Suppose $X$ is a random variable and $\Psi$ is a nonnegative increasing function on $[0, \infty)$. For each $\varepsilon > 0$, show that $P\{|X| \geq \varepsilon\} \leq P\Psi(|X|)/\Psi(\varepsilon)$.

11. Let $X_1, \ldots, X_n$ be i.i.d. with $P\{X_i = 1\} = p$ and $P\{X_i = 0\} = 1 - p$. Define $Y_i = X_i - p$. Note that $X = X_1 + \ldots + X_n$ has a $\text{Bin}(n, p)$ distribution. (i) Find $P(Y_i)$ and $P(Y_i^2)$ and $P(Y_i^4)$. (ii) Find $P(\Sigma_{i=1}^n Y_i)^2$ and $P(\Sigma_{i=1}^n Y_i)^4$. (iii) Show that $P\{|X - np| \geq \varepsilon\} \leq P(\Sigma_{i=1}^n Y_i)^2/\varepsilon^2$ and $P\{|X - np| \geq \varepsilon\} \leq P(\Sigma_{i=1}^n Y_i)^4/\varepsilon^4$. (iv) For two bounds in (iii), which one is better?

12. Suppose $X$ and $Y$ are independent random variables, with $X \sim \text{Poisson}(\lambda)$ and $Y \sim \text{Poisson}(\mu)$. Show that given $X + Y = n$, the random variable $X$ has a $\text{Bin}(n, p)$ distribution with $p = \lambda/(\lambda + \mu)$.

13. Suppose $X$ has a continuous distribution with density $f(\cdot)$. Let $Y = a + bX$ and $Z = -X$, where $a$ and $b$ are constants with $b > 0$. Find the density functions for the distributions of $Y$ and $Z$.

14. Let $Z_1, Z_2, \ldots, Z_n$ be i.i.d. $N(0, 1)$. Let $Y = \sqrt{\frac{Z_1^2 + \ldots + Z_n^2}{n-1}}$. What is the density of $Y$?

15. Let $X_1, \ldots, X_n$ be i.i.d. exponential($\lambda$). Let $M = \min\{X_1, \ldots, X_n\}$. Find the density for $M$.

16. Suppose $Z$ has a $N(0, 1)$ distribution. Define $\mu_k = PZ^k$ for each nonnegative integer $k$. Note that $\mu_0 = 1$. (i) Explain why $\mu_k = 0$ when $k$ is odd. (ii) Find the moment generating function $g_Z(t) = Pe^{t^2}$ of $Z$. (iii) Find $\mu_4$. 

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17. Suppose $X$ and $Y$ are independent random variables, each with density function $f(t) = e^{-t}$ for $t > 0$. Let $U = X$ and $V = X - Y$. Write $\Psi$ for the joint density function of $U$ and $V$. (i) Indicate the region of the $(U,V)$-plane where $\Psi > 0$. (ii) Find $\Psi$. (iii) Find the distribution of $X - Y$. 